

H_{ar}^1 for Arithmetic Surface is Finite

K. Sugahara, L. Weng

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Abstract

For an arithmetic surface X and a Weil divisor D , there are natural arithmetic cohomology groups $H_{\text{ar}}^i(X, \mathcal{O}_X(D))$ ($i = 0, 1, 2$). Using ind-pro topology on adelic space $\mathbb{A}_{X,012}^{\text{ar}}$, we show that $H_{\text{ar}}^0(X, \mathcal{O}_X(D))$ is discrete, $H_{\text{ar}}^1(X, \mathcal{O}_X(D))$ is finite, and $H_{\text{ar}}^2(X, \mathcal{O}_X(D))$ is compact. Moreover, we prove that all possible summations of canonical subspaces $\mathbb{A}_{X,i}^{\text{ar}}(D)$, $\mathbb{A}_{X,kl}^{\text{ar}}(D)$ ($i, k, l = 0, 1, 2$) are closed in $\mathbb{A}_{X,012}^{\text{ar}}$, and hence complete our proof of topological dualities of among H_{ar}^i 's.

1 Arithmetic Cohomology Groups

In this section, we review natural constructions and basic properties involving various canonical adelic spaces $\mathbb{A}_{X,*}^{\text{ar}}(D)$'s and arithmetic cohomology groups $H_{\text{ar}}^i(X, \mathcal{O}_X(D))$ ($i = 0, 1, 2$) associated to an arithmetic surface X and a Weil divisor D . For details, please refer to [2].

Let $\pi : X \rightarrow \text{Spec}(\mathcal{O}_F)$ be an arithmetic surface over integer ring \mathcal{O}_F of a number field F . We assume that π is projective and flat, that X is integral and regular. Denote by X_F its generic fiber, and $k(X)$, resp. $k(X_F)$, the field of rational functions of X , resp. of X_F . We have $k(X) = k(X_F)$.

Let (X, C, x) be a flag of X , consisting of an integral curve C on X and a closed point x of C . Denote by $k(X)_{C,x}$ its local ring. It is known that $k(X)_{C,x}$ is a direct sum of two dimensional local fields. Since X is a Noetherian scheme, following Parshin-Beilinson ([4], [1]), we have a two dimensional adelic space $\mathbb{A}_{X,012}^{\text{fin}}$, its level two subspaces $\mathbb{A}_{X,01}^{\text{fin}}$, $\mathbb{A}_{X,02}^{\text{fin}}$, $\mathbb{A}_{X,12}^{\text{fin}}(D)$ and the associated level one subspaces $\mathbb{A}_{X,0}^{\text{fin}}$, $\mathbb{A}_{X,1}^{\text{fin}}(D)$, $\mathbb{A}_{X,01}^{\text{fin}}(D)$, for a Weil divisor D on X . These spaces can be roughly described as follows:

$$\begin{aligned} \mathbb{A}_X^{\text{fin}} &= \mathbb{A}_{X,012}^{\text{fin}} := \mathbb{A}_{X,012}^{\text{fin}}(\mathcal{O}_X) := \prod'_{(C,x): x \in C} k(X)_{C,x} := \prod'_C \left(\prod'_{x: x \in C} k(X)_{C,x} \right), \\ \mathbb{A}_{X,01}^{\text{fin}} &:= \{(f_C)_{C,x} \in \mathbb{A}_{X,012}\}, \quad \mathbb{A}_{X,02}^{\text{fin}} := \{(f_x)_{C,x} \in \mathbb{A}_{X,012}\}, \\ \mathbb{A}_{X,12}^{\text{fin}}(D) &:= \{(f_{C,x})_{C,x} \in \mathbb{A}_{X,012} \mid \text{ord}_C(f_{C,x}) + \text{ord}_C(D) \geq 0 \ \forall C \subset X\}, \\ \mathbb{A}_{X,0}^{\text{fin}} &:= \mathbb{A}_{X,01}^{\text{fin}} \cap \mathbb{A}_{X,02}^{\text{fin}}, \quad \mathbb{A}_{X,1}^{\text{fin}}(D) := \mathbb{A}_{X,01}^{\text{fin}} \cap \mathbb{A}_{X,12}^{\text{fin}}(D), \quad \mathbb{A}_{X,2}^{\text{fin}}(D) := \mathbb{A}_{X,02}^{\text{fin}} \cap \mathbb{A}_{X,12}^{\text{fin}}(D). \end{aligned}$$

It is well-known that $\mathbb{A}_{X,012}$ admits a natural ind-pro structure

$$\mathbb{A}_{X,012}^{\text{fin}}(\mathcal{O}_X) = \varinjlim_{D_1} \varprojlim_{D_2: D_2 \leq D_1} \mathbb{A}_{X,12}(D_1) / \mathbb{A}_{X,12}(D_1).$$

Furthermore, following [3], introduce the adelic space at infinity by

$$\mathbb{A}_X^\infty := \mathbb{A}_{X_F} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varinjlim_{D_1} \varprojlim_{D_2: D_2 \leq D_1} \left((\mathbb{A}_{X_F,1}(D_1) / \mathbb{A}_{X_F,1}(D_1)) \otimes_{\mathbb{Q}} \mathbb{R} \right),$$

and define

$$\mathbb{A}_X^{\text{ar}} := \mathbb{A}_{X,012}^{\text{ar}} := \mathbb{A}_X^{\text{fin}} \bigoplus \mathbb{A}_X^\infty.$$

Then $\mathbb{A}_{X,012}^{\text{ar}}$ admits three level two subspaces $\mathbb{A}_{X,01}^{\text{ar}}, \mathbb{A}_{X,02}^{\text{ar}}, \mathbb{A}_{X,12}^{\text{ar}}(D)$ ([2, §2.3.1]), which can be described by

$$\begin{aligned} \mathbb{A}_{X,01}^{\text{ar}} &= \{ (f_{C,x}) \times (f_P) \in \mathbb{A}_X^{\text{ar}} \mid (f_{C,x})_{C,x} = (f_C)_{C,x} \in \mathbb{A}_{X,01}^{\text{fin}}, f_{E_P} = f_P \ \forall P \in X_F \}, \\ \mathbb{A}_{X,02}^{\text{ar}} &= \mathbb{A}_{X,02}^{\text{fin}} \bigoplus k(X_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}, \quad \mathbb{A}_{X,12}^{\text{ar}}(D) := \mathbb{A}_{X,12}^{\text{fin}}(D) \bigoplus (\mathbb{A}_{X_F,1}(D_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}). \end{aligned}$$

Here, as usual, within the adelic space $\mathbb{A}_{X_F,01}$ for the curve X_F/F , we have

$$\mathbb{A}_{X_F,1}(D_F) := \{ (f_P) \in \mathbb{A}_{X_F,01} \mid \text{ord}_P(f_P) + \text{ord}_P(D_F) \geq 0 \ \forall P \in X_F \}$$

and we, moreover, define

$$\mathbb{A}_{X_F,1}(D_F) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} := \varprojlim_{E_F: E_F \leq D_F} \left(\mathbb{A}_{X_F,1}(D_F) / \mathbb{A}_{X_F,1}(E_F) \otimes_{\mathbb{Q}} \mathbb{R} \right),$$

using the natural diagonal imbeddings $k(X) \hookrightarrow \mathbb{A}_{X_F} \hookrightarrow \mathbb{A}_X^{\text{ar}}$. Similarly, we have three level one canonical subspaces

$$\mathbb{A}_{X,0}^{\text{ar}} := \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}}, \quad \mathbb{A}_{X,1}^{\text{ar}}(D) := \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D), \quad \mathbb{A}_{X,2}^{\text{ar}}(D) := \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D).$$

Accordingly, there are natural ind-pro structures on $\mathbb{A}_{X,012}^{\text{ar}}$ ([2, §2.4.3]):

$$\mathbb{A}_X^{\text{ar}} = \varinjlim_D \varprojlim_{E: E \leq D} \mathbb{A}_{X,12}^{\text{ar}}(D) / \mathbb{A}_{X,12}^{\text{ar}}(E),$$

and on $\mathbb{A}_{X,01}^{\text{ar}}$ and $\mathbb{A}_{X,02}^{\text{ar}}$ ([2, Corollary 14]):

$$\begin{aligned} \mathbb{A}_{X,01}^{\text{ar}} &= \varinjlim_D \varprojlim_{E: E \leq D} \mathbb{A}_{X,1}^{\text{ar}}(D) / \mathbb{A}_{X,1}^{\text{ar}}(E), \\ \mathbb{A}_{X,02}^{\text{ar}} &= \varinjlim_D \varprojlim_{E: E \leq D} \mathbb{A}_{X,2}^{\text{ar}}(D) / \mathbb{A}_{X,2}^{\text{ar}}(E). \end{aligned}$$

Based on these genuine ind-pro structures on arithmetic adelic spaces, we may introduce natural ind-pro topologies on these spaces, as what we do for locally compact spaces. With topologies hence defined, following [2, §§3.1.2-3], particularly, [2, Proposition 26], we know that natural inclusions of the above canonical level two and hence level one subspaces to the total space are all continuous with closed images. Consequently, these subspaces are also Hausdorff, since $\mathbb{A}_{X,012}^{\text{ar}}$ is Hausdorff by [2, Theorem II]. For later references, we summarize this as

Proposition 1. ([2, Proposition 26]) *Let D be a Weil divisor on X . Its canonical adelic spaces $\mathbb{A}_{01}^{\text{ar}}, \mathbb{A}_{02}^{\text{ar}}, \mathbb{A}_{12}^{\text{ar}}(D), \mathbb{A}_0^{\text{ar}}, \mathbb{A}_1^{\text{ar}}(D), \mathbb{A}_2^{\text{ar}}(D)$ are closed and Hausdorff.*

We introduce arithmetic cohomology groups as follows:

Definition 1. ([2, §2.4.1]) *Let D be a Weil divisor on X . Then we define arithmetic cohomology groups $H_{\text{ar}}^i(X, D)$ of D on X ($i = 0, 1, 2$) by*

$$\begin{aligned} H_{\text{ar}}^0(X, D) &:= \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D); \\ H_{\text{ar}}^1(X, D) &:= \\ &= ((\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}) \cap \mathbb{A}_{X,12}^{\text{ar}}(D)) / (\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) + \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)); \\ H_{\text{ar}}^2(X, D) &:= \mathbb{A}_{X,012}^{\text{ar}} / (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D)). \end{aligned}$$

In the sequel, to simplify notations, when write $\mathbb{A}_{X,*}^{\text{ar}}$, we will omit X . For example, we write $\mathbb{A}_{X,012}^{\text{ar}}$ simply as $\mathbb{A}_{012}^{\text{ar}}$.

2 Adelic Sub-Quotient Spaces

Lemma 2. *Let E be a Weil divisor on X . Then we have*

- (1) $\mathbb{A}_{01}^{\text{ar}} / \mathbb{A}_1(E)^{\text{ar}}$ is discrete;
- (2) $\mathbb{A}_{12}^{\text{ar}}(E) / \mathbb{A}_1(E)^{\text{ar}}$ is compact.

Proof. (1) Using inductive limit, we have

$$\mathbb{A}_{01}^{\text{ar}} / \mathbb{A}_1(E)^{\text{ar}} = \varinjlim_{D: D \geq E} \mathbb{A}_1^{\text{ar}}(D) / \mathbb{A}_1(E)^{\text{ar}}$$

where D runs over all Weil divisors on X such that $D - E$ is effective. Note that if $D = E + C$ for an integral curve on X , either horizontal or vertical, we have $\mathbb{A}_1^{\text{ar}}(D) / \mathbb{A}_1(E)^{\text{ar}} \simeq k(C) = \mathbb{A}_{C,0}^{\text{ar}}$, which is discrete (say, in $\mathbb{A}_{C,01}^{\text{ar}}$). Hence, for any $D \geq E$, the quotient space $\mathbb{A}_1^{\text{ar}}(D) / \mathbb{A}_1(E)^{\text{ar}}$ is discrete as there exist finitely many C_i 's such that $D - E = \sum_i C_i$. To complete proof, recall that topologies on our adelic spaces are induced from the ind-pro one, that is, the strongest topology making all inductive limits continuous. Consequently, as an inductive limit of discrete spaces, $\mathbb{A}_{01}^{\text{ar}} / \mathbb{A}_1(E)^{\text{ar}}$ is discrete.

(2) From exact sequence

$$\mathbb{A}_1^{\text{ar}}(E) / \mathbb{A}_1^{\text{ar}}(E') \rightarrow \mathbb{A}_{12}^{\text{ar}}(E) / \mathbb{A}_{12}^{\text{ar}}(E') \rightarrow \frac{\mathbb{A}_{12}^{\text{ar}}(E) / \mathbb{A}_{12}^{\text{ar}}(E')}{\mathbb{A}_1^{\text{ar}}(E) / \mathbb{A}_1^{\text{ar}}(E')}, \quad (*)$$

by taking projective limit on E' , we obtain an exact sequence

$$\mathbb{A}_1^{\text{ar}}(E) \rightarrow \mathbb{A}_{12}^{\text{ar}}(E) \rightarrow \mathbb{A}_{12}^{\text{ar}}(E) / \mathbb{A}_1^{\text{ar}}(E).$$

(One may also see this using first $\frac{\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_{12}^{\text{ar}}(E')}{\mathbb{A}_1^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E')} \simeq \frac{\mathbb{A}_{12}^{\text{ar}}(E)}{\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_{12}^{\text{ar}}(E')}$ and then $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E) = \varprojlim_{E': E' \leq E} \frac{\mathbb{A}_{12}^{\text{ar}}(E)}{\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_{12}^{\text{ar}}(E')}$.) Hence to prove $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact, being the projective limit, it suffices to show that the quotient spaces $\frac{\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_{12}^{\text{ar}}(E')}{\mathbb{A}_1^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E')}$ are compact. On the other hand, as above, for any integral curve C on X , there is a canonical exact sequence

$$0 \rightarrow \mathbb{A}_{C,0}^{\text{ar}} \rightarrow \mathbb{A}_{C,01}^{\text{ar}} \rightarrow \mathbb{A}_{C,01}^{\text{ar}}/\mathbb{A}_{C,0}^{\text{ar}} \rightarrow 0 \quad (**)$$

with $\mathbb{A}_{C,01}^{\text{ar}}/\mathbb{A}_{C,0}^{\text{ar}}$ compact. That is to say, $\frac{\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_{12}^{\text{ar}}(E-C)}{\mathbb{A}_1^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E-C)}$ is compact, since exact sequences $(*)$ and $(**)$ are equivalent when $E = E' + C$. Consequently, $\frac{\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_{12}^{\text{ar}}(E')}{\mathbb{A}_1^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E')}$ are compact for any Weil divisors $E \geq E'$, by first writing $E - E' = \sum_i C_i \geq 0$, then arguing in the same way as in proof of (1) above. \square

Proposition 3. *Let E be a Weil divisor on X . Then as subspaces of $\mathbb{A}_{012}^{\text{ar}}$, we have*

- (1) $\mathbb{A}_{01}^{\text{ar}}(E) + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed;
- (2) $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)$ is closed;
- (3) $\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E)$ is closed.

Proof. (1) Recall that $\mathbb{A}_1(E)^{\text{ar}} := \mathbb{A}_{01}^{\text{ar}} \cap \mathbb{A}_{12}^{\text{ar}}(E)$, and by Lemma 2(2), the quotient $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact. Hence, its topological dual is discrete and given by $\frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}((\omega) - E)}{\mathbb{A}_{12}^{\text{ar}}((\omega) - E)}$. As a subspace, $\frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}((\omega) - E)}{\mathbb{A}_{12}^{\text{ar}}((\omega) - E)}$ is then discrete. Since $\mathbb{A}_{12}^{\text{ar}}((\omega) - E)$ is closed, and all quotient spaces involved here are Hausdorff, $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}((\omega) - E)$ is closed. So is $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$, since E is arbitrary above.

(2) Similarly, by Lemma 2(1), the quotient $\mathbb{A}_{01}^{\text{ar}}/\mathbb{A}_1^{\text{ar}}(E)$ is discrete. Hence $\mathbb{A}_{01}^{\text{ar}}/(\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E))$ is discrete. Clearly, it is also Hausdorff. Thus with the fact that $\mathbb{A}_{01}^{\text{ar}}$ is closed, we have $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)$ is closed.

(3) By definition, $H_{\text{ar}}^0(X, \mathcal{O}_X(E)) = \mathbb{A}_{01}^{\text{ar}} \cap \mathbb{A}_{02}^{\text{ar}} \cap \mathbb{A}_{12}^{\text{ar}}(E) = (\mathbb{A}_{01}^{\text{ar}} \cap \mathbb{A}_{12}^{\text{ar}}(E)) \cap (\mathbb{A}_{02}^{\text{ar}} \cap \mathbb{A}_{12}^{\text{ar}}(E)) = \mathbb{A}_1^{\text{ar}}(E) \cap \mathbb{A}_2^{\text{ar}}(E)$. Hence there exists a natural surjection $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E)) \twoheadrightarrow (\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E))/\mathbb{A}_1^{\text{ar}}(E)$. By Lemma 4 below, $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is compact. So is $(\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E))/\mathbb{A}_1^{\text{ar}}(E)$. It is also Hausdorff, since $\mathbb{A}_1^{\text{ar}}(E)$ is closed and $\mathbb{A}_{012}^{\text{ar}}$ is Hausdorff. Therefore, $\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E)$ is closed; \square

Lemma 4. *For a Weil divisor E on X , $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is compact.*

Proof. This can be deduced from the natural exact sequence

$$0 \rightarrow \frac{H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E))} \rightarrow \frac{\mathbb{A}_2^{\text{ar}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E))} \rightarrow \frac{\mathbb{A}_2^{\text{ar}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)} \rightarrow 0.$$

Indeed, $\frac{H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E))}$ is nothing but $\mathbb{A}_2^{\text{fin}}(E)$, since, by definition, $H_{\text{ar}}^0(X, \mathcal{O}_X(E)) \cap \mathbb{A}_2^{\text{fin}}(E) = \{0\}$. Moreover, $\mathbb{A}_2^{\text{fin}}(E) = \varprojlim_E \frac{\mathbb{A}_2^{\text{fin}}(E)}{\mathbb{A}_2^{\text{fin}}(E')}$. Thus,

by the fact that, for an integral curve C on X , $\frac{\mathbb{A}_2^{\text{fin}}(E)}{\mathbb{A}_2^{\text{fin}}(E-C)} \simeq \mathbb{A}_{C,1}^{\text{fin}}(E|_C)$ is compact, we conclude that $\frac{\mathbb{A}_2^{\text{fin}}(E)}{\mathbb{A}_2^{\text{fin}}(E')}$ and hence $\frac{H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E))}$ are compact as well. On the other hand, by definition, $\mathbb{A}_{X,2}^{\text{ar}}(E) = \mathbb{A}_{X,2}^{\text{fin}}(E) + H^0(X, \mathcal{O}_X(E)) \otimes_{\mathbb{Z}} \mathbb{R}$ and $H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E) = H^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)$. Hence, we have $\frac{\mathbb{A}_2^{\text{ar}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E)) + \mathbb{A}_2^{\text{fin}}(E)} = \frac{\mathbb{A}_{X,2}^{\text{fin}}(E) + H^0(X, \mathcal{O}_X(E)) \otimes_{\mathbb{Z}} \mathbb{R}}{\mathbb{A}_2^{\text{fin}}(E) + H^0(X, \mathcal{O}_X(E))}$ is naturally isomorphic to $\frac{H^0(X, \mathcal{O}_X(E)) \otimes_{\mathbb{Z}} \mathbb{R}}{H^0(X, \mathcal{O}_X(E))}$, a compact torus. \square

Theorem 5. *For a Weil divisor D on X , $H_{\text{ar}}^1(X, \mathcal{O}_X(D))$ is finite.*

Proof. By proof of Proposition 3(2), $\frac{\mathbb{A}_{01}^{\text{ar}} \cap (\mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D))}{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(D)} = H_{\text{ar}}^1(X, \mathcal{O}_X(D))$, is discrete. On the other hand, $\frac{\mathbb{A}_{12}^{\text{ar}}(D)}{\mathbb{A}_1(D)^{\text{ar}} + \mathbb{A}_2(D)^{\text{ar}}}$ is compact, since, by Lemma 2(2), $\mathbb{A}_{12}^{\text{ar}}(D)/\mathbb{A}_1(D)^{\text{ar}}$ is compact. By Proposition 3(3), resp. Proposition 1, $\mathbb{A}_1(D)^{\text{ar}} + \mathbb{A}_2(D)^{\text{ar}}$, resp. $\mathbb{A}_{12}^{\text{ar}}(D)$, is closed. So $\frac{\mathbb{A}_{12}^{\text{ar}}(D)}{\mathbb{A}_1(D)^{\text{ar}} + \mathbb{A}_2(D)^{\text{ar}}}$ is Hausdorff, since $\mathbb{A}_{01}^{\text{ar}}$ is Hausdorff. By Proposition 3(1), $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)$ is closed. Hence $\frac{\mathbb{A}_{12}^{\text{ar}}(D) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D))}{\mathbb{A}_1(D)^{\text{ar}} + \mathbb{A}_2(D)^{\text{ar}}}$ is closed and hence compact. But this later sub-quotient is another expression of $H_{\text{ar}}^1(X, \mathcal{O}_X(D))$. Hence, $H_{\text{ar}}^1(X, \mathcal{O}_X(D))$ is both compact and discrete. Therefore, it is finite. \square

Corollary 6. *For a Weil divisor E on X , $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_2^{\text{ar}}(E)$ is closed.*

Proof. Note that $H^1(X, \mathcal{O}_X(E))$ can also be written as $\frac{\mathbb{A}_{02}^{\text{ar}} \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))}{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_2^{\text{ar}}(E)^{\text{ar}}}$. By Proposition 3(1), $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed. Hence, the numerator is also closed. But H_{ar}^1 is finite and Hausdorff, hence the denominator, namely, $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_2^{\text{ar}}(E)$ is closed. \square

Lemma 7. *Let E be a Weil divisor on X . Then we have*

- (1) $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed;
- (2) $\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_{02}^{\text{ar}}$ is closed;
- (3) $\mathbb{A}_2^{\text{ar}}(E) + \mathbb{A}_{01}^{\text{ar}}$ is closed.

Proof. (1) By Lemma 2(2), $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact. With the natural surjection $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E) \rightarrow \frac{\mathbb{A}_{12}^{\text{ar}}(E) + \mathbb{A}_0^{\text{ar}}}{\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_0^{\text{ar}}}$, the later quotient space is also compact. Clearly, it is also Hausdorff. By Proposition 3(2), the denominator $\mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_0^{\text{ar}}$ is closed. Hence the numerator $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed as well.

(2) Since $\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}} = \lim_{D' \uparrow} \lim_{E: E \leq D'} \frac{(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(D')}{(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E)}$, it suffices to show

that $\frac{(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(D')}{(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E)}$, or more strongly, $(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E)$ is closed.

We first assume that $E \geq D$. In this case, $(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E) = \mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_2^{\text{ar}}(E)$. By Lemma 4, $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is compact. This implies that $\frac{\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_2^{\text{ar}}(E)}{\mathbb{A}_1^{\text{ar}}(D) + H_{\text{ar}}^0(X, \mathcal{O}_X(E))}$ is compact, since there is a natural surjection

$\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E)) \rightarrow \frac{\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_2^{\text{ar}}(E)}{\mathbb{A}_1^{\text{ar}}(D) + H_{\text{ar}}^0(X, \mathcal{O}_X(E))}$. Thus, it suffices to show that $\mathbb{A}_1^{\text{ar}}(D) + H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is closed. Now, by Lemma 2(2), the quotient space $\mathbb{A}_{12}^{\text{ar}}(D)/\mathbb{A}_1^{\text{ar}}(D)$ is compact. Hence,

$\frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)}{(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))}$ is compact, from surjection $\mathbb{A}_{12}^{\text{ar}}(D)/\mathbb{A}_1^{\text{ar}}(D) \rightarrow \frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)}{(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))}$.

Furthermore, with the closeness of the spaces $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)$, resp. $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$, as proved in Proposition 3(1), resp. Proposition 8(3) below, whose proof is independent of this argument, we conclude that this latest quotient space $\frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)}{(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))}$ is nothing but the topological dual of $(\mathbb{A}_1^{\text{ar}}(D) + H^0(X, \mathcal{O}_X(E)))/\mathbb{A}_1^{\text{ar}}(D)$. So $(\mathbb{A}_1^{\text{ar}}(D) + H^0(X, \mathcal{O}_X(E)))/\mathbb{A}_1^{\text{ar}}(D)$ is discrete. Therefore, $\mathbb{A}_1^{\text{ar}}(D) + H^0(X, \mathcal{O}_X(E))$ is closed, as required.

To complete the proof, we next treat general E . Fix a Weil divisor D' on X such that $D' \geq D$, $D' \geq E$. By the special case just proved, we have $(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(D')$ is closed. Since $(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E) = ((\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(D')) \cap \mathbb{A}_{12}^{\text{ar}}(E)$, $(\mathbb{A}_1^{\text{ar}}(D) + \mathbb{A}_{02}^{\text{ar}}) \cap \mathbb{A}_{12}^{\text{ar}}(E)$ is closed.

(3) By Lemma 4, $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is compact. With the natural surjection $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E)) \rightarrow (\mathbb{A}_2^{\text{ar}}(E) + \mathbb{A}_{01}^{\text{ar}})/\mathbb{A}_{01}^{\text{ar}}$, this latest space is also compact. Clearly, it is Hausdorff as well. Consequently, with denominator being closed, the numerator $\mathbb{A}_2^{\text{ar}}(E) + \mathbb{A}_{01}^{\text{ar}}$ is closed. \square

Proposition 8. *Let E be a Weil divisor on X . Then we have*

- (1) $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}}$ is closed;
- (2) $\mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed;
- (3) $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed;
- (4) $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E)$ is closed.

Proof. (1) By Theorem 5, $H^1(X, \mathcal{O}_X(E)) = \frac{\mathbb{A}_{12}^{\text{ar}}(E) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})}{\mathbb{A}_1(E)^{\text{ar}} + \mathbb{A}_2(E)^{\text{ar}}}$, is finite. Hence, the numerator $\mathbb{A}_{12}^{\text{ar}}(E) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))$ is closed since the denominator $\mathbb{A}_1(E)^{\text{ar}} + \mathbb{A}_2(E)^{\text{ar}}$ is closed by Proposition 3(3). Consequently, for any two Weil divisors D, E with $D \geq E$, viewing as subspaces of $\mathbb{A}_{12}^{\text{ar}}(D)/\mathbb{A}_{12}^{\text{ar}}(E)$, the space $\frac{\mathbb{A}_{12}^{\text{ar}}(D) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})}{\mathbb{A}_{12}^{\text{ar}}(E) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})}$ is closed. On the other hand, by definition,

$$\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} = \lim_{\rightarrow D} \lim_{\leftarrow E: E \leq D} \frac{\mathbb{A}_{12}^{\text{ar}}(D) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})}{\mathbb{A}_{12}^{\text{ar}}(E) \cap (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})}.$$

Therefore, by definition of ind-pro topology, the subspace $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}}$ is closed.

(2) By Lemma 2(2), $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact. With natural surjection $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E) \rightarrow \frac{\mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)}{\mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)}$, the latest space is compact as well. Thus as above, since, by Lemma 7(2), the denominator is closed, the numerator $\mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed.

(3) By Lemma 2(2), $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact. With natural surjection $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E) \rightarrow \frac{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)}{\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)}$, the latest space is compact as well. But $\mathbb{A}_{01}^{\text{ar}} \supset \mathbb{A}_1^{\text{ar}}(E)$, hence the denominator is simply $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}}$. By (1), it is closed. Consequently, as above, the numerator $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed.

(4) By Lemma 4, $\mathbb{A}_2^{\text{ar}}(E)/H_{\text{ar}}^0(X, \mathcal{O}_X(E))$ is compact. By definition, we have $H_{\text{ar}}^0(X, \mathcal{O}_X(E)) = \mathbb{A}_0^{\text{ar}} \cap \mathbb{A}_1^{\text{ar}}(E) \cap \mathbb{A}_2^{\text{ar}}(E)$. Hence, from composition of natural surjections $\frac{\mathbb{A}_2^{\text{ar}}(E)}{H_{\text{ar}}^0(X, \mathcal{O}_X(E))} \rightarrow \frac{\mathbb{A}_2^{\text{ar}}(E)}{(\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)) \cap \mathbb{A}_2^{\text{ar}}(E)} \rightarrow \frac{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}} + \mathbb{A}_2^{\text{ar}}(E)}{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)}$,

we conclude that $\frac{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E)}{\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)}$ is compact. By Proposition 3(2), $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E)$ is closed. Consequently, $\mathbb{A}_0^{\text{ar}} + \mathbb{A}_1^{\text{ar}}(E) + \mathbb{A}_2^{\text{ar}}(E)$ is closed as well. \square

Theorem 9. *Let D be a Weil divisor on X . Then we have*

- (1) \mathbb{A}_0^{ar} is discrete. In particular, $H_{\text{ar}}^0(X, \mathcal{O}_X(D))$ is discrete;
- (2) $H^2(X, \mathcal{O}_X(D))$ is compact.

Proof. (1) By Lemma 2(2), $\mathbb{A}_{12}^{\text{ar}}(E)/\mathbb{A}_1^{\text{ar}}(E)$ is compact. Taking its topological dual, we have $(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))/\mathbb{A}_{12}^{\text{ar}}(E)$ is discrete, since $\mathbb{A}_1^{\text{ar}}(E) = \mathbb{A}_{01}^{\text{ar}} \cap \mathbb{A}_{12}^{\text{ar}}(E)$ and, by Proposition 3(1), $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E)$ is closed. Consequently, $(\mathbb{A}_0^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))/\mathbb{A}_{12}^{\text{ar}}(E)$ is discrete, since we have a natural injection $(\mathbb{A}_0^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))/\mathbb{A}_{12}^{\text{ar}}(E) \hookrightarrow (\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))/\mathbb{A}_{12}^{\text{ar}}(E)$. In particular, $\mathbb{A}_0^{\text{ar}}/H^0(X, \mathcal{O}_X(E))$ is discrete since, by the second isomorphism theorem, there is a natural isomorphism $(\mathbb{A}_0^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(E))/\mathbb{A}_{12}^{\text{ar}}(E) \simeq \mathbb{A}_0^{\text{ar}}/H^0(X, \mathcal{O}_X(E))$. By definition, $H^0(X, \mathcal{O}_X(E)) \subset H^0(X_F, \mathcal{O}_{X_F}(E_F))$, where $\mathcal{O}_{X_F}(E_F)$ denotes the restriction of $\mathcal{O}_X(E)$ to the generic fiber X_F of arithmetic surface $X \rightarrow \text{Spec } \mathcal{O}_F$. Hence, we may choose E with negative enough $\mathcal{O}_{X_F}(E_F)$ so

as to get a vanishing $H^0(X, \mathcal{O}_X(E))$. This then implies that \mathbb{A}_0^{ar} and hence also $H^0(X, \mathcal{O}_X(D))$ are discrete.

(2) Taking topological dual, we see that $\mathbb{A}_{012}^{\text{ar}}/(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}})$ is compact, since, by Proposition 8(1), $\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}}$ is closed. This then implies that $H^2(X, \mathcal{O}_X(D))$ is compact, since we have a natural surjection $\mathbb{A}_{012}^{\text{ar}}/(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}}) \rightarrow \mathbb{A}_{012}^{\text{ar}}/(\mathbb{A}_{01}^{\text{ar}} + \mathbb{A}_{02}^{\text{ar}} + \mathbb{A}_{12}^{\text{ar}}(D)) = H^2(X, \mathcal{O}_X(D))$. \square

3 Topological Duality of $H_{\text{ar}}^i(X, \mathcal{O}_X(D))$'s

In this section, we use adelic theory for arithmetic surfaces developed in [2] and closeness of various natural subspaces of $\mathbb{A}_X^{\text{ar}} = \mathbb{A}_{012}^{\text{ar}}$ proved in the previous section to establish the following fundamental

Theorem 10. *Let X be an arithmetic surface with a canonical divisor K_X and D be a Weil divisor on X . Then we have the following topological dualities*

$$H_{\text{ar}}^i(\widehat{X, \mathcal{O}_X(D)}) \simeq H_{\text{ar}}^{2-i}(X, \mathcal{O}_X(K_X - D)) \quad i = 0, 1, 2.$$

Remark. A proof of this theorem is stated in [2, §3.2.4]. However, a key condition on closeness of two sum spaces used is omitted in a well-known lemma cited there, e.g. Lemma 12 below. This was pointed out by Osipov at the end of January 2016. Now with such closeness confirmed in §2, we can complete our proof.

Recall that there is a natural global residue pairing on $\mathbb{A}_{X,012}^{\text{ar}}$, introduced using local residue theory. Indeed, by [2, §2.2], for a fixed non-zero rational differential ω on X , we can define a global pairing on $\mathbb{A}_{X,012}^{\text{ar}}$ with respect to ω by

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\omega} : \quad \mathbb{A}_{X,012}^{\text{ar}} \times \mathbb{A}_{X,012}^{\text{ar}} &\longrightarrow \mathbb{S}^1 \\ ((f_{C,x}, f_{P,\sigma}), (g_{C,x}, g_{P,\sigma})) &\mapsto \sum_{C \subset X, x \in C: \pi(x) \in S_{\text{fin}}} \text{Res}_{C,x}(f_{C,x} g_{C,x} \omega) \\ &\quad + \sum_{P \in X_F} \sum_{\sigma \in S_{\infty}} \text{Res}_{P,\sigma}(f_{P,\sigma} g_{P,\sigma} \omega). \end{aligned}$$

Here $\text{Res}_{C,x}$, resp. $\text{Res}_{P,\sigma}$, denotes the local pairing on X , resp. on X_{σ} . Moreover, we have

Proposition 11. *Let X be an arithmetic surface, D a Weil divisor and ω a non-zero rational differential on X . Then we have*

- (1) ([2, Lemma 11]) *The pairing $\langle \cdot, \cdot \rangle_{\omega}$ is well defined;*
- (2) ([2, Proposition 12]) *The pairing $\langle \cdot, \cdot \rangle_{\omega}$ is non-degenerate;*
- (3) ([2, Proposition 15]) *With respect to $\langle \cdot, \cdot \rangle_{\omega}$, we have*

$$(\mathbb{A}_{X,01}^{\text{ar}})^{\perp} = \mathbb{A}_{X,01}^{\text{ar}}, \quad (\mathbb{A}_{X,02}^{\text{ar}})^{\perp} = \mathbb{A}_{X,02}^{\text{ar}}, \quad (\mathbb{A}_{X,12}^{\text{ar}}(D))^{\perp} = \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D).$$

To apply this result, we use the following well-known

Lemma 12. *With respect to a continue, non-degenerate pairing $\langle \cdot, \cdot \rangle$ on a topological space \mathcal{W} , e.g. $\mathbb{A}_{X,012}^{\text{ar}}$, we have*

- (1) *If W_1 and W_2 are closed subgroups of \mathcal{W} such that the spaces $W_1 + W_2$ and $W_1^\perp + W_2^\perp$ are closed, then*

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp \quad \text{and} \quad (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp;$$

- (2) *If W is a closed subgroup of \mathcal{W} , then, algebraically and topologically,*

$$(W^\perp)^\perp = W \quad \text{and} \quad W \simeq \widehat{\mathcal{W}/W^\perp}.$$

With above preparation, we are ready to prove our theorem.

Proof. (A) Topological duality between H_{ar}^0 and H_{ar}^2

By definition of H_{ar}^2 in §2.1, Lemma 12, and Proposition 11(3), we have

$$\begin{aligned} H_{\text{ar}}^2(\widehat{X, (\omega) - D}) &\simeq \left(\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D) \right)^\perp \\ &\simeq \left(\mathbb{A}_{X,01}^{\text{ar}} \right)^\perp \cap \left(\mathbb{A}_{X,02}^{\text{ar}} \right)^\perp \cap \left(\mathbb{A}_{X,12}^{\text{ar}}((\omega) - D) \right)^\perp \\ &= \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) \simeq H_{\text{ar}}^0(D). \end{aligned}$$

Indeed, by Proposition 8(1), $\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}$ is closed, and by Proposition 8(3), $\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)$ is closed.

(B) Topological duality among H_{ar}^1

For H_{ar}^1 , similarly as in [4], see [2, Proposition 16], we have the following group theoretic isomorphisms:

$$\begin{aligned} &H_{\text{ar}}^1(X, D) \\ &\simeq \left(\mathbb{A}_{X,01}^{\text{ar}} \cap (\mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D)) \right) / \left(\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) \right) \\ &\simeq \left(\mathbb{A}_{X,02}^{\text{ar}} \cap (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D)) \right) / \left(\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) \right). \end{aligned}$$

Consequently, we have

$$\begin{aligned} H_{\text{ar}}^1(\widehat{X, (\omega) - D}) &= \left(\frac{\mathbb{A}_{X,02}^{\text{ar}} \cap (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))}{\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)} \right)^\perp \\ &\simeq \frac{(\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,02}^{\text{ar}})^\perp \cap (\mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))^\perp}{(\mathbb{A}_{X,02}^{\text{ar}})^\perp + (\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D))^\perp} \\ &= \frac{(\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}) \cap (\mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D))}{\mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)} \\ &\simeq \frac{(\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}) \cap \mathbb{A}_{X,12}^{\text{ar}}(D)}{\mathbb{A}_{X,01}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D) + \mathbb{A}_{X,02}^{\text{ar}} \cap \mathbb{A}_{X,12}^{\text{ar}}(D)} \simeq H_{\text{ar}}^1(X, D). \end{aligned}$$

Indeed, the first equality and the last isomorphism are direct consequence of the definition. To verify the second isomorphism and the third equality, we use Proposition 8(1) and (2), namely, $\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}}$ and $\mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}(D)$ are closed, and Proposition 8(3), namely, $\mathbb{A}_{X,01}^{\text{ar}} + \mathbb{A}_{X,02}^{\text{ar}} + \mathbb{A}_{X,12}^{\text{ar}}((\omega) - D)$ is closed. As for the forth morphisms, both associated quotients spaces are Hausdorff, since denominators and numerators are all closed and $\mathbb{A}_{X,012}^{\text{ar}}$ is Hausdorff. They are also discrete since H_{ar}^1 is always so. Therefore, this forth morphism is a topological homeomorphism, being a group isomorphism. \square

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K. Sugahara & L. Weng
Graduate School of Mathematics,
Kyushu University,
Fukuoka, 819-0395,
Japan
E-mails: k-sugahara@math.kyushu-u.ac.jp,
weng@math.kyushu-u.ac.jp